



Adaptive posicast controller for time-delay systems with relative degree $n^* \leq 2^*$

Yildiray Yildiz^{a,*}, Anuradha Annaswamy^a, Ilya V. Kolmanovsky^b, Diana Yanakiev^b

^a Active Adaptive Control Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

^b Research and Innovation Center, Ford Motor Company, 2101 Village Road, Dearborn, MI 48121, USA

ARTICLE INFO

Article history:

Received 27 April 2009

Received in revised form

29 July 2009

Accepted 2 November 2009

Available online 30 November 2009

Keywords:

Adaptive control

Time-delay systems

Model matching

ABSTRACT

In this paper, we present an Adaptive Posicast Controller that deals with parametric uncertainties in linear systems with delays. It is assumed that the plant has no right half plane zeros and the delay is known. The adaptive controller is based on the Smith Predictor and Finite Spectrum Assignment with time-varying parameters adjusted online. A novel Lyapunov–Krasovskii functional is used to show semi-global stability of the closed-loop error equations. The controller is applied to engine fuel-to-air ratio control. The implementation results show that the Adaptive Posicast Controller significantly improves the closed-loop performance when compared to the case with the existing baseline controller.

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1. Introduction

Time-delays are ubiquitous in dynamical systems, present as computational delays, input delays, measurement delays and transportation/convection lags to name a few. Often, higher order dynamical systems can be modeled as a low order system together with a time-delay. Examples of time-delay systems can be found in a wide range of applications including chemical, biological, mechanical, physiological and electrical systems (see Kolmanovsky & Myshkis, 1999; Niculescu, 2001; Wang, Lee, & Tan, 1999). Detailed surveys of time-delay systems can be found in Richard (2003) and Gu and Niculescu (2003).

The control of systems involving time-delays is a difficult problem since the existence of a delay may induce instability or bad performance for the closed-loop system. In many controller designs, the delay is neglected and stability and robustness margins are given with respect to delay. The same approach can also be found in some adaptive control designs (Jang, 2008). However, in general, these approaches may produce small delay margins.

One of the most popular methods for controlling systems with time-delays is the Smith Predictor (SP), proposed by Otto Smith in

the 1950s (Smith, 1959). The main idea in this approach is predicting the future output of the plant, using a plant model, and using this prediction to cancel the effect of the time-delay (Astrom, Hang, & Lim, 1994; Fliess, Marquez, & Mounier, 2001; Majhi & Atherton, 1998; Palmor, 1996; Smith, 1976). This future prediction inspired the name for the “posicast”, which stands for “positively casting”, or “forecasting” (Park, Wee, & Annaswamy, 2002). The SP method, however, is not suitable for unstable systems due to the possibility of unstable pole-zero cancellations. In Manitius and Olbrot (1979) a method based on finite spectrum assignment (FSA) was introduced by Manitius and Olbrot that allows control of unstable systems with time-delay, where the main idea was using finite time integrals to prevent unstable pole-zero cancellations (see Glusing-Luerben, 1997; Loiseau & Brethe, 1998; Watanabe, Nobuyama, Kitamori, & Ito, 1992, for other variations of FAS methods). A combination of the FSA together with a pole-placement based control design was proposed in Ichikawa (1985), and an adaptive version of the same was proposed in Ortega and Lozano (1988) where the augmented error approach proposed in Narendra and Annaswamy (2005) was used. The approach in Ortega and Lozano (1988) guarantees a stable adaptive system for any linear time-invariant plant with parametric uncertainties, stable zeros and a known order, time-delay, and upper bound on the relative degree.

In Niculescu and Annaswamy (2003), the control of time-delay systems whose relative degree does not exceed two was addressed. It was shown that a simple adaptive law, similar to that in the time-delay free case (Narendra & Annaswamy, 2005), can be used to lead to stability. In Niculescu and Annaswamy (2003), as in Ichikawa (1985), Ortega and Lozano (1988), and Narendra and Annaswamy (2005), the plant poles are restricted to multiplicity one. In addition, the underlying Lyapunov function arguments

^{*} The material in this paper was partially presented at ASME Dynamic Systems and Control Conference, Ann Arbor, Michigan, 2008. This paper was recommended for publication in revised form by Associate Editor Changyun Wen under the direction of Editor Miroslav Krstic.

* Corresponding address: U.C. Santa Cruz, UARC, NASA Ames Research Center, Moffett Field, 94035 CA, USA. Tel.: +1 650 6044382; fax: +1 650 604 4036.

E-mail addresses: yildiray.yildiz@nasa.gov (Y. Yildiz), aanna@mit.edu (A. Annaswamy), ikolmano@ford.com (I.V. Kolmanovsky), dyanakie@ford.com (D. Yanakiev).

deploy structures that depend on finite-dimensional state space that may lead to additional restrictions on the locations of the plant poles and zeros. There also exist studies for unknown time-delay (Wen, Soh, & Zhang, 2000) where the delay is approximated using rational functions, which leads to a fairly complex controller structure and an asymptotic tracking error although stability is guaranteed.

In this paper, we provide an adaptive control design for plants with a relative degree less than or equal to two with unknown parameters and a known time-delay. No restrictions on the pole locations are imposed, and the zeros are assumed to be stable. Unlike the control structure in Ichikawa (1985), Ortega and Lozano (1988), and Niculescu and Annaswamy (2003), we derive a controller using predictions of future states, which gives a better insight into its design. A novel Lyapunov–Krasovskii functional that only consists of partial states of the overall infinite-dimensional system is used to establish stability. This is carried out at the expense of a slightly more complex set of induction based arguments. The overall control design is shown to remain simple, despite its ability to stabilize a larger class of plants with time-delays.

The simplicity of the underlying control design together with its adaptive feature has led to a very successful implementation in several full-scale experimental studies (Yildiz, Annaswamy, Yanakiev, & Kolmanovsky, 2007, 2008a,b) in automotive applications that pertained to different problems in powertrain systems with large time-delays. For the sake of completeness, a brief description of the results obtained is included in this paper.

The main contributions of this paper can be summarized as follows: (i) Extension of the existing adaptive control results for linear uncertain plants with time-delays in Ichikawa (1985), Ortega and Lozano (1988) and Niculescu and Annaswamy (2003) to the cases when (a) the locations of the poles have no restrictions, which is enabled via the construction of a Lyapunov functional that only depends on the finite-dimensional states, and (b) the multiplicity of the poles can be arbitrary, which is enabled via the next contribution; (ii) a transparent derivation of the control architecture using a design that predicts the states τ s in the future compared to those in Ichikawa (1985), Ortega and Lozano (1988) and Niculescu and Annaswamy (2003), thereby allowing a simpler and more intuitive design procedure (see Mirkin & Raskin, 2003a, for a similar result); (iii) lower control order and significantly simpler adaptive laws compared to the results in Ichikawa (1985) and Ortega and Lozano (1988).

The organization of the paper is as follows: In Section 2, we explain the APC design for time-delay systems when the plant is first order. In Section 3, we give the APC design for systems with relative degree smaller than or equal to two, where we have only input–output measurements. In this section, we first give a simple model matching controller design for time-delay systems and later we introduce the APC building upon this model matching controller. We give the stability proof by induction, employing Lyapunov–Krasovskii functionals. In Section 4, we present the results of experiments where we applied the APC to fuel-to-air ratio control problem, using a test vehicle.

2. First order plant

We begin with a simple problem, where the plant is given by

$$\dot{x}(t) = ax(t) + u(t) \quad (1)$$

for which a control input of the form

$$u(t) = \theta^*x(t) + r(t) \quad \theta^* = a_m - a, \quad a_m < 0 \quad (2)$$

ensures stable tracking for any a . One can provide a more formal guarantee of such a tracking by choosing a reference model of the form

$$\dot{x}_m(t) = a_mx_m(t) + r(t) \quad (3)$$

which leads to error dynamics of the form

$$\dot{e}(t) = a_me(t) \quad e(t) = x(t) - x_m(t) \quad (4)$$

for which it can be simply shown that $V(t) = \frac{1}{2}e^2(t)$ is a Lyapunov function, with $\dot{V} < 0$, leading to exponential stability.

2.1. The posicast controller

We now introduce a time-delay in (1) so that

$$\dot{x}(t) = ax(t) + u(t - \tau) \quad (5)$$

where the goal is to stabilize the plant and track the output of a stable reference model. The results of Smith (1959) and Manitius and Olbrot (1979) inspire us to establish the following approach: A posicast controller that “positively” forecasts the output is chosen as

$$u(t) = \theta^*x(t + \tau) + r(t) \quad (6)$$

which in turn leads to a closed-loop system of the form

$$\dot{x}(t) = a_mx(t) + r(t - \tau), \quad (7)$$

an obviously stable plant. The non-causal controller in (6) can be shown to be indeed causal with a clever algebraic manipulation established in Manitius and Olbrot (1979). This is enabled by observing that the plant equation in (5) can be written in an integral form as

$$x(t + \tau) = e^{a\tau}x(t) + \int_{-\tau}^0 e^{-a\eta}u(t + \eta)d\eta, \quad (8)$$

leading to a causal controller determined by (6) together with (8). The above observation also leads us to a posicast Lyapunov function for the closed-loop system given by (5)–(6), for the case when $r(t) = 0$, given by

$$V(t) = \frac{1}{2}x^2(t + \tau). \quad (9)$$

It can be shown from (8) and some algebra that

$$\dot{V}(t) = a_mx^2(t + \tau). \quad (10)$$

2.2. The adaptive posicast controller

We now proceed to the case when a is unknown. Using customary adaptive control procedures (Narendra & Annaswamy, 2005), suppose we choose a control input of the form

$$u(t) = \theta x(t + \tau) + r(t) \quad (11)$$

it leads to a closed-loop system of the form

$$\begin{aligned} \dot{x}(t) = a_mx(t) + \tilde{\theta}(t - \tau) & \left[e^{a\tau}x(t - \tau) \right. \\ & \left. + \int_{-\tau}^0 e^{-a\eta}u(t + \eta - \tau)d\eta \right] + r(t - \tau) \end{aligned} \quad (12)$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$. While indeed this suggests that a reference model can be chosen in the form

$$\dot{x}_m(t) = a_mx_m(t) + r(t - \tau), \quad (13)$$

it can be seen that it poses a difficulty, since the underlying error model can be derived using (12) and (13) as

$$\begin{aligned} \dot{e}(t) = a_me(t) + \tilde{\theta}(t - \tau) \\ \times \left[e^{a\tau}x(t - \tau) + \int_{-\tau}^0 e^{-a\eta}u(t + \eta - \tau)d\eta \right]. \end{aligned} \quad (14)$$

Eq. (14) is however not in a form that lends itself to a Lyapunov function since the term inside the square brackets includes the unknown parameter a . We therefore choose a different control input that is still motivated by the non-adaptive controller given in (6).

Eqs. (6) and (8) imply that the posicast control input is essentially of the form

$$u(t) = \theta_x^* x(t) + \int_{-\tau}^0 \lambda^*(\eta) u(t + \eta) d\eta + r(t) \quad (15)$$

where

$$\theta_x^* = \theta^* e^{a\eta}, \quad \lambda^*(\eta) = \theta^* e^{a\eta}.$$

Therefore, a choice of a control input of the form

$$u(t) = \theta_x(t) x(t) + \int_{-\tau}^0 \lambda(t, \eta) u(t + \eta) d\eta + r(t) \quad (16)$$

leads to a closed-loop system of the form

$$\begin{aligned} \dot{x}(t) = & ax(t) + \theta^* \left[e^{a\tau} x(t - \tau) \right. \\ & \left. + \int_{-\tau}^0 e^{-a\eta} u(t - \tau + \eta) d\eta \right] + \tilde{\theta}_x(t - \tau) x(t - \tau) \\ & + \int_{-\tau}^0 \tilde{\lambda}(t - \tau, \eta) u(t - \tau + \eta) d\eta + r(t - \tau) \end{aligned} \quad (17)$$

where

$$\tilde{\theta}_x(t) = \theta_x(t) - \theta_x^*, \quad \tilde{\lambda}(t, \eta) = \lambda(t, \eta) - \lambda^*(\eta).$$

From (8), it follows that (17) can be written as

$$\begin{aligned} \dot{x}(t) = & a_m x(t) + \tilde{\theta}_x(t - \tau) x(t - \tau) \\ & + \int_{-\tau}^0 \tilde{\lambda}(t - \tau, \eta) u(t - \tau + \eta) d\eta + r(t - \tau). \end{aligned} \quad (18)$$

As a result, defining $e(t) = x(t) - x_m(t)$, (18) and (13) imply that the underlying error model is of the form

$$\begin{aligned} \dot{e}(t) = & a_m e(t) + \tilde{\theta}_x(t - \tau) x(t - \tau) \\ & + \int_{-\tau}^0 \tilde{\lambda}(t - \tau, \eta) u(t - \tau + \eta) d\eta. \end{aligned} \quad (19)$$

This error model is further discussed in Section 3. Here, we give the stability result and leave the proof to Section 3, where the general case, n th order system, is investigated.

Theorem 1. *Given initial conditions $\tilde{\theta}_x(0)$, $x(\xi)$, $\tilde{\lambda}(\xi, \eta)$ for $\xi \in [-\tau, 0]$ and $u(\zeta)$ for $\zeta \in [-2\tau, 0]$, there exists a τ^* such that for all $\tau \in [0, \tau^*]$, the plant in (1), controller in (16), and adaptive laws given by*

$$\begin{aligned} \dot{\tilde{\theta}}_x(t) = & -\gamma_1 e(t) x(t - \tau) \\ \frac{\partial \lambda}{\partial t}(t, \eta) = & -\gamma_2 e(t) u(t + \eta - \tau) \end{aligned} \quad (20)$$

have bounded solutions for all $t \geq 0$ and $\lim_{t \rightarrow \infty} e(t) = 0$.

It is noted that for a higher order plant with input time-delay where the finite-dimensional state variables are accessible, the controller design can be accomplished following similar lines as above.

3. Adaptive posicast control in the case of output measurements with $n^* \leq 2$

In this section, the focus is on higher order time-delay systems with relative degree, n^* , smaller than two.

3.1. Exact model matching for delayed systems

Consider the plant with the time-delay τ whose input–output description is given as

$$y(t) = W_p(s) u(t - \tau), \quad W_p(s) = k_p \frac{Z_p(s)}{R_p(s)} \quad (21)$$

where s denotes the differential operator d/dt , $Z_p(s)$ and $R_p(s)$ are monic coprime polynomials with order m and n and $n^* = n - m > 0$ is defined as the relative order of the finite-dimensional part of the plant. It is also assumed that $Z_p(s)$ is Hurwitz and k_p is a constant gain parameter. The reference input–output description is given by

$$y_m(t) = W_m(s) r(t - \tau), \quad W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} \quad (22)$$

where $Z_m(s)$ and $R_m(s)$ are monic Hurwitz polynomials of degrees m_m and n_m respectively, and k_m is a constant gain parameter. Further, it is assumed that $n_m - m_m \geq n - m$.

The model matching problem is to determine a bounded control input $u(t)$ to the plant such that the closed-loop transfer function of the plant together with the controller, from $r(t)$ to $y(t)$, matches the reference model transfer function.

Consider the following state space representation of the plant dynamics (21), together with two “signal generators” formed by a controllable pair Λ, l :

$$\dot{x}_p(t) = A_p x_p(t) + b_p u(t - \tau), \quad y(t) = h_p^T x_p(t) \quad (23)$$

$$\dot{\omega}_1(t) = \Lambda \omega_1(t) + l u(t - \tau) \quad (24)$$

$$\dot{\omega}_2(t) = \Lambda \omega_2(t) + l y(t) \quad (25)$$

where, $\Lambda \in \mathbb{R}^{n \times n}$ and $l \in \mathbb{R}^n$. Defining $\bar{x}(t) \triangleq x(t + \tau)$, $\bar{\omega}_i(t) \triangleq \omega_i(t + \tau)$, $i = 1, 2$, $\bar{y}(t) \triangleq y(t + \tau)$, (23)–(25) can be rewritten in the following form:

$$\dot{\bar{x}}_p(t) = A_p \bar{x}_p(t) + b_p u(t), \quad \bar{y}(t) = h_p^T \bar{x}_p(t) \quad (26)$$

$$\dot{\bar{\omega}}_1(t) = \Lambda \bar{\omega}_1(t) + l u(t) \quad (27)$$

$$\dot{\bar{\omega}}_2(t) = \Lambda \bar{\omega}_2(t) + l \bar{y}(t). \quad (28)$$

Under our assumptions, it follows (Narendra & Annaswamy, 2005) that there exists $\beta_1^*, \beta_2^* \in \mathbb{R}^n$ and $k^* \in \mathbb{R}$ such that the control law

$$u(t) = \beta_1^{*T} \bar{\omega}_1(t) + \beta_2^{*T} \bar{\omega}_2(t) + k^* r(t) \quad (29)$$

results in the exact model matching of the reference model (22):

$$\frac{\bar{y}(t)}{r(t)} = k_m \frac{Z_m(s)}{R_m(s)}. \quad (30)$$

It is noted that the model matching is achieved between the rational parts of the closed-loop transfer function and of the reference model. On the other hand, it follows directly from (30) that

$$\frac{y(t)}{r(t)} = k_m \frac{Z_m(s)}{R_m(s)} e^{-\tau s} \quad (31)$$

which shows that the exact model matching condition is also satisfied between the total closed loop and the reference model transfer functions. Since $\bar{\omega}_2(t)$ requires the output measurement at time $t + \tau$, the control law given in (29) is non-causal. We refer to this closed-loop system with the non-causal controller as “Controller NC” and it is depicted in Fig. 1.

It is also well known (Narendra & Annaswamy, 2005) that with zero initial conditions, there exists $c, d \in \mathbb{R}^n$ such that $\bar{y}(t) = c^T \bar{\omega}_1(t) + d^T \bar{\omega}_2(t)$, which implies

$$y(t) = c^T \omega_1(t) + d^T \omega_2(t). \quad (32)$$

Substituting (32) into (25) and rewriting (24) and (25) together in vector-matrix form, we get

$$\begin{pmatrix} \dot{\omega}_1(t) \\ \dot{\omega}_2(t) \end{pmatrix} = A \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} + b u(t - \tau)$$

$$\text{where, } A = \begin{bmatrix} \Lambda & 0 \\ lc^T & \Lambda + ld^T \end{bmatrix} \text{ and } b = \begin{bmatrix} l \\ 0 \end{bmatrix}. \quad (33)$$

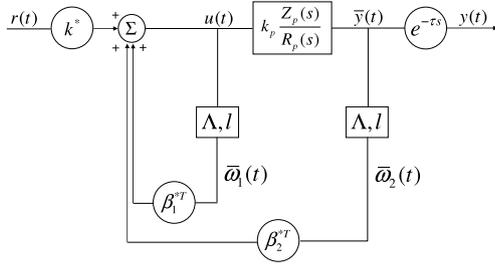


Fig. 1. Exact model matching, Controller NC.

From (33) we get

$$\begin{pmatrix} \bar{\omega}_1(t) \\ \bar{\omega}_2(t) \end{pmatrix} = e^{A\tau} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} + \int_{-\tau}^0 e^{-A\sigma} b u(t + \sigma) d\sigma. \quad (34)$$

Substituting (34) into (29) we obtain that

$$u(t) = \{[\beta_1^{*T} \quad \beta_2^{*T}] e^{A\tau}\} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} + \int_{-\tau}^0 \{[\beta_1^{*T} \quad \beta_2^{*T}] e^{-A\sigma} b\} u(t + \sigma) d\sigma + k^* r(t), \quad (35)$$

which can equivalently be expressed as

$$u(t) = \alpha_1^{*T} \omega_1(t) + \alpha_2^{*T} \omega_2(t) + \int_{-\tau}^0 \phi^*(\sigma) u(t + \sigma) d\sigma + k^* r(t). \quad (36)$$

The controller (36) is causal and we refer to it as ‘‘Controller C’’. Defining $\beta^* = [\beta_1^{*T} \quad \beta_2^{*T}]$ and $\alpha^* = [\alpha_1^{*T} \quad \alpha_2^{*T}]$, we get the following relationships between the controller parameters of Controller NC and Controller C

$$\alpha^* = \beta^* e^{A\tau} \quad (37)$$

$$\phi^*(\sigma) = \beta^* e^{-A\sigma} b. \quad (38)$$

The block diagram of Controller C is shown in Fig. 2.

In summary, an exact model matching controller, Controller C, can be designed for the time-delay system (21) using a well known procedure for the delay free systems (Narendra & Annaswamy, 2005). It is noted that the equivalence between Controller NC and Controller C shows that the latter relies on predicting the future output, $\bar{y}(t)$ to achieve exact model matching. This can be viewed as an equivalence result for higher order systems similar to that shown in Section 2 between (11) and (15).

It is noted that the model matching controller (36) was already introduced in Ichikawa (1985) and Manitius and Olbrot (1979). The main contribution here is that we derive this controller by using the delay free part of the system dynamics, which considerably simplifies the design. In addition, instead of proposing a controller structure and proving that it is a model matching controller, we derive the controller using future state prediction which gives an insight into the controller structure. Finally, this derivation removes the assumption that the plant has poles with multiplicity one, which was assumed in the earlier papers. A similar result is given in Mirkin and Raskin (2003b).

3.2. Adaptive controller

The goal of this section is to design a model reference adaptive controller, motivated by the equivalence of the Controller NC and Controller C, for the plant described in (21) with unknown coefficients for the polynomials $Z_p(s)$ and $R_p(s)$ and with unknown, positive high frequency gain k_p .

3.2.1. Design for $n^* = 1$

In this section, we assume that the relative degree n^* of the plant is one. The first step of the adaptive controller design is to

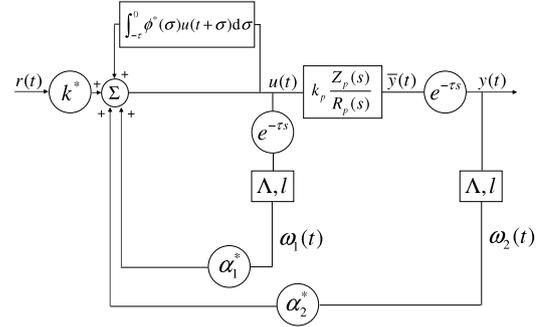


Fig. 2. Exact model matching, Controller C.

design a fixed controller which satisfies the exact model matching with the reference model assuming that the plant parameters are known. This step was already completed in Section 3.1 resulting in the controller given in (36). In the second step, the controller parameters $\alpha_1^*, \alpha_2^*, \phi^*(\sigma)$ and k^* are replaced by time-varying parameters $\alpha_1(t), \alpha_2(t), \phi(t, \sigma)$ and $k(t)$ and the goal is to determine the laws by which they are adjusted so as to result in a stable system.

We define the following parameters:

$$\begin{aligned} \tilde{k}(t) &\triangleq k(t) - k^*, & \tilde{\alpha}_1(t) &\triangleq \alpha_1(t) - \alpha_1^*, \\ \tilde{\alpha}_2(t) &\triangleq \alpha_2(t) - \alpha_2^*, & \tilde{\phi}(t, \sigma) &\triangleq \phi(t, \sigma) - \phi^*(\sigma), \\ \alpha(t) &\triangleq [\alpha_1^T(t) \quad \alpha_2^T(t)]^T, & \beta(t) &\triangleq [\beta_1^T(t) \quad \beta_2^T(t)]^T, \\ \Omega(t) &\triangleq [\omega_1^T(t) \quad \omega_2^T(t) \quad r(t)]^T, & \omega(t) &\triangleq [\omega_1^T(t) \quad \omega_2^T(t)]^T, \\ \theta(t) &\triangleq [\alpha_1^T(t) \quad \alpha_2^T(t) \quad k(t)]^T, & \tilde{\theta}(t) &\triangleq \theta(t) - \theta^*, \\ \theta^* &\triangleq [(\alpha_1^*)^T \quad (\alpha_2^*)^T \quad k^*]^T. \end{aligned} \quad (39)$$

Consider a controller

$$\begin{aligned} u(t) &= \alpha^T(t) \omega(t) + \int_{-\tau}^0 \phi(t, \sigma) u(t + \sigma) d\sigma + k(t) r(t) \\ &= \alpha_1^{*T} \omega_1(t) + \alpha_2^{*T} \omega_2(t) + \int_{-\tau}^0 \phi^*(\sigma) u(t + \sigma) d\sigma \\ &\quad + k^* r(t) + \tilde{\alpha}_1^T(t) \omega_1(t) + \tilde{\alpha}_2^T(t) \omega_2(t) \\ &\quad + \int_{-\tau}^0 \tilde{\phi}(t, \sigma) u(t + \sigma) d\sigma + \tilde{k}(t) r(t). \end{aligned} \quad (40)$$

Using the equivalence between (29) and (36), we can rewrite (40) as

$$\begin{aligned} u(t) &= \beta_1^{*T} \bar{\omega}_1(t) + \beta_2^{*T} \bar{\omega}_2(t) + k^* r(t) + \tilde{\alpha}_1^T(t) \omega_1(t) \\ &\quad + \tilde{\alpha}_2^T(t) \omega_2(t) + \int_{-\tau}^0 \tilde{\phi}(t, \sigma) u(t + \sigma) d\sigma + \tilde{k}(t) r(t). \end{aligned} \quad (41)$$

The differential equations, describing the plant (23) together with the controller (41) can then be represented as

$$\begin{aligned} \dot{X}_p(t) &= A_m X_p(t) + b_m \left[\tilde{\theta}^T(t - \tau) \Omega(t - \tau) \right. \\ &\quad \left. + \int_{-\tau}^0 \tilde{\phi}(t - \tau, \sigma) u(t - \tau + \sigma) d\sigma + k^* r(t - \tau) \right], \\ y_p(t) &= h_m^T X_p(t) \end{aligned} \quad (42)$$

where

$$A_m \triangleq \begin{bmatrix} A_p & b_p \beta_1^{*T} & b_p \beta_2^{*T} \\ 0 & \Lambda + l \beta_1^{*T} & l \beta_2^{*T} \\ l h_p^T & 0 & \Lambda \end{bmatrix}, \quad b_m \triangleq \begin{bmatrix} b_p \\ l \\ 0 \end{bmatrix},$$

$$X_p(t) \triangleq [x_p^T(t) \ \omega_1^T(t) \ \omega_2^T(t)]^T, \quad h_m^T \triangleq [h_p^T \ 0 \ 0],$$

$$y_p = y. \tag{43}$$

We showed in Section 3.1 that when the parameter errors are equal to zero, the closed-loop transfer function is identical to that of the reference model. Therefore, the reference model can be described by the (3n)th order differential equation

$$\dot{X}_m(t) = A_m X_m(t) + b_m k^* r(t - \tau),$$

$$y_m(t) = h_m^T X_m(t) \tag{44}$$

where,

$$X_m(t) \triangleq [x_p^{*T} \ \omega_1^{*T} \ \omega_2^{*T}]^T,$$

$$h_m^T (sI - A_m)^{-1} b_m k^* = k_m \frac{Z_m}{R_m}. \tag{45}$$

It is noted that $x_p^*(t)$, $\omega_1^*(t)$ and $\omega_2^*(t)$ can be considered as the signals in the reference model corresponding to $x_p(t)$, $\omega_1(t)$ and $\omega_2(t)$ in the closed-loop system. Therefore, subtracting (44) from (42), we get an error equation for the overall system as

$$\dot{e}(t) = A_m e(t) + b_m \left[\tilde{\theta}^T(t - \tau) \Omega(t - \tau) + \int_{-\tau}^0 \tilde{\phi}(t - \tau, \sigma) u(t - \tau + \sigma) d\sigma \right],$$

$$e_1(t) = h_m^T e(t), \tag{46}$$

where $e(t) = X_p - X_m$ and $e_1(t) = y_p(t) - y_m(t)$.

The adaptive control law which guarantees convergence of the tracking error to zero and boundedness of all signals has the following form:

$$\dot{\tilde{\theta}}(t) = -\gamma_\theta e_1(t) \Omega(t - \tau)$$

$$\dot{\tilde{\phi}}(t, \sigma) = -\gamma_\phi e_1(t) u(t - \tau + \sigma), \quad -\tau \leq \sigma \leq 0 \tag{47}$$

where, γ_θ and γ_ϕ are positive, real adaptation gains.

3.2.2. Design for $n^* = 2$

When the relative degree n^* is equal to unity, it is easy to define a strictly positive real reference model $W_m(s)$. When $n^* = 2$, an addition of an input u_a to u as

$$u_a = \hat{\theta}^T \Omega' + \int_{-\tau}^0 \hat{\phi}(t, \sigma) u'(t + \sigma) d\sigma,$$

$$\dot{\Omega}' = -a \Omega' + \Omega,$$

$$\dot{u}' = -a u' + u, \quad a > 0 \tag{48}$$

can be used to derive yet another error equation of the form

$$e(t) = W_m(s)(s + a)e^{-st}$$

$$\times \left[\tilde{\theta}^T(t) \Omega(t) + \int_{-\tau}^0 \tilde{\phi}(t, \sigma) u(t + \sigma) d\sigma \right] \tag{49}$$

where $a > 0$ is chosen such that $(s + a)W_m(s)$ is strictly positive real. Therefore it suffices to consider the stability of (46). The results can then be extended to the case when $n^* = 2$ by making use of the additional input u_a .

3.3. Stability analysis

The following theorem confirms the desirable properties of the adaptive control law (47).

Theorem 2. Given initial conditions $x_p(0)$, $u(\eta)$, $\eta \in [t_0 - 2\tau, t_0]$, $\tilde{\theta}(\xi)$, $\tilde{\phi}(\xi)$, $\xi \in [t_0 - \tau, t_0]$, $\exists \tau^* \text{ s.t. } \forall \tau \in [0, \tau^*]$, the plant (21), the controller (40), and the adaptive laws given by (47) have bounded solutions $\forall t \geq 0$ and $\lim_{t \rightarrow \infty} e(t) = 0$, $\lim_{t \rightarrow \infty} e_1(t) = 0$.

We first state and prove the following lemma.

Lemma 1. Suppose a continuous function $u(t)$ satisfies

$$u(t) = f(t) + \int_{-\tau}^0 \phi(t, \sigma) u(t + \sigma) d\sigma \tag{50}$$

where $u, f : [t_0 - \tau, \infty) \rightarrow \mathfrak{R}$, $\phi : [t_0, \infty) \times [-\tau, 0] \rightarrow \mathfrak{R}$. Then,

$$|u(t'_j)| \leq 2(\bar{f} + c_1 c_0) e^{c_0(t'_j - t'_i)}, \quad \forall t'_j \geq t'_i, \tag{51}$$

if constants $t'_i, \bar{f}, c_0, c_1 \in \mathfrak{R}^+$ exist such that $\|f(t)\| \leq \bar{f}$,

$$\int_{-\tau}^0 \phi^2(t, \sigma) d\sigma \leq c_0^2 \quad \text{for } t \in [t'_i, t'_j], \tag{52}$$

and

$$\int_{-\tau}^0 u^2(t + \sigma) d\sigma \leq c_1^2 \quad \forall t \leq t'_i. \tag{53}$$

Proof of Lemma 1. Since (50) is an integral equation with a square integrable kernel, it has a unique solution. We derive inequality (51) by considering a sequence u_n , so that u is the limit of this sequence as $n \rightarrow \infty$ (Debnath & Mikusinski, 2005).

Define

$$u_0(t'_j) \triangleq u(t'_j), \quad t'_j < t'_i$$

$$\triangleq 0, \quad t'_j \geq t'_i \tag{54}$$

$$u_{n+1}(t'_j) \triangleq u(t'_j), \quad t'_j < t'_i,$$

$$\triangleq \int_{-\tau}^0 \phi(t'_j, \sigma) u_n(t'_j + \sigma) d\sigma + f(t'_j), \quad t'_j \geq t'_i,$$

$$n = 0, 1, \dots, \infty. \tag{55}$$

For $t'_j \geq t'_i$ and $n = 1$, we have that

$$|u_1(t'_j) - u_0(t'_j)| = \left| f(t'_j) + \int_{-\tau}^0 \phi(t'_j, \sigma) u_0(t'_j + \sigma) d\sigma \right|$$

$$\leq \left(\int_{-\tau}^0 (\phi(t'_j, \sigma))^2 d\sigma \right)^{1/2} \left(\int_{-\tau}^0 (u_0(t'_j + \sigma))^2 d\sigma \right)^{1/2} + \bar{f}. \tag{56}$$

Using (52) and (53), the inequality (56) can be rewritten as

$$|u_1(t'_j) - u_0(t'_j)| \leq \bar{f} + c_0 c_1. \tag{57}$$

For $t'_j \geq t'_i$ and $n = 2$ we have that

$$|u_2(t'_j) - u_1(t'_j)| = \left| \int_{-\tau}^0 \phi(t'_j, \sigma) (u_1(t'_j + \sigma) - u_0(t'_j + \sigma)) d\sigma \right|$$

$$\leq \left(\int_{-\tau}^0 \phi^2(t'_j, \sigma) \right)^{1/2}$$

$$\times \left(\int_{-\tau}^0 |u_1(t'_j + \sigma) - u_0(t'_j + \sigma)|^2 d\sigma \right)^{1/2}. \tag{58}$$

Using inequality (52) and a change of variables $\zeta = t + \sigma$, (58) becomes

$$|u_2(t'_j) - u_1(t'_j)| \leq c_0 \left(\int_{t'_j - \tau}^{t'_j} |u_1(\zeta) - u_0(\zeta)|^2 d\zeta \right)^{1/2}$$

$$= c_0 \left(\int_{t'_j - \tau}^{t'_j} |u_1(\zeta) - u_0(\zeta)|^2 d\zeta + \int_{t'_i}^{t'_j} |u_1(\zeta) - u_0(\zeta)|^2 d\zeta \right)^{1/2}. \quad (59)$$

We know from (54) and (55) that $u_1(\zeta) - u_0(\zeta) = 0$ for $\zeta < t'_i$. Therefore, (59) can be further simplified as

$$|u_2(t'_j) - u_1(t'_j)| \leq c_0 \left(\int_{t'_i}^{t'_j} |u_1(\zeta) - u_0(\zeta)|^2 d\zeta \right)^{1/2}. \quad (60)$$

Substituting (57) into (60) we obtain that

$$|u_2(t'_j) - u_1(t'_j)| \leq (\bar{f} + c_1 c_0) c_0 \sqrt{(t'_j - t'_i)}. \quad (61)$$

Continuing this procedure iteratively, we obtain that

$$|u_{n+1}(t'_j) - u_n(t'_j)| \leq (\bar{f} + c_1 c_0) \cdot \left(\frac{(c_0^2 (t'_j - t'_i))^n}{n!} \right)^{1/2}. \quad (62)$$

It is noted that the term $(t'_j - t'_i)^n/n!$ in (62) is obtained due to successive integrations of $(t'_j - t'_i)$. It can be shown using the ratio test (Rudin et al., 1976) that the series $\sum_{n=1}^{\infty} (\bar{f} + c_0 c_1) \left(\frac{(c_0^2 (t - t'_i))^n}{n!} \right)^{1/2}$ converges. This in turn implies that the series

$$S = \sum_{n=1}^{\infty} (u_n - u_{n-1}) \quad (63)$$

is absolutely convergent and it can be shown (see Appendix) that

$$S(t'_j) \leq 2(\bar{f} + c_1 c_0) e^{c_0^2 (t'_j - t'_i)}. \quad (64)$$

From (63), $\lim_{n \rightarrow \infty} u_n$ exists and is equal to u per (55). Then, (64) implies that

$$|u(t'_j) - u_0(t'_j)| \leq 2(\bar{f} + c_1 c_0) e^{c_0^2 (t'_j - t'_i)}. \quad (65)$$

This implies that,

$$|u(t'_j)| \leq 2(\bar{f} + c_1 c_0) e^{c_0^2 (t'_j - t'_i)} \quad t'_j \geq t'_i \quad (66)$$

and this completes the proof. Next, we present the proof of Theorem 2. □

Proof of Theorem 2. The proof is by induction. Let the statement S be given by

$$S : \|X_p(\xi)\| \leq I_0, \quad |u(\xi)| \leq U(I_0) \quad \forall \xi \in [t_0, t_0 + k\tau] \quad (67)$$

where $U(\cdot)$ is an analytic, bounded function of its arguments.

It is noted that $X_p(\xi)$ and $u(\xi)$ is bounded for $\xi \in [t_0 - 2\tau, t_0]$. Using this fact, we complete the proof by showing that

- I. S is true for $k = 1$.
- II. If S is true for k , then it is true for $k + 1$.

I and II allow us to conclude that all signals are bounded for all $t \geq t_0$. Finally, from Barbalat's lemma, convergence of the error e to zero follows.

3.3.1. I. S is true for $k = 1$

The proof of I is given in four steps, each of which starts with a brief summary.

Step 1. Some bounds on the signals are assumed in the time interval $[t_0 - 2\tau)$ and the negative semi-definiteness of the Lyapunov functional time derivative in $[t_0, t_0 + \tau)$ is shown, which yields the boundedness of the signals in $[t_0, t_0 + \tau)$. In addition, using Lemma 1, an upper bound for the control signal $u(t)$ in $[t_0, t_0 + \tau)$ is given.

Consider a Lyapunov–Krasovskii functional

$$V(t) = e(t)^T P e(t) + \tilde{\theta}(t)^T \tilde{\theta}(t) + \int_{-\tau}^0 \tilde{\phi}(t, \sigma)^2 d\sigma + \int_{-\tau}^0 \int_{t+\nu}^t \dot{\tilde{\theta}}(\xi)^T \dot{\tilde{\theta}}(\xi) d\xi d\nu + \int_{-\tau}^0 \int_{t+\nu}^0 \int_{-\tau}^0 (\dot{\tilde{\phi}}(\xi, \sigma))^2 d\sigma d\xi d\nu \quad (68)$$

where $P > 0$. The error model (46) and the Lyapunov functional (68) has been discussed in Niculescu and Annaswamy (2003). After algebraic manipulations, upper bound on the Lyapunov functional derivative can be computed as follows:

$$\dot{V}(t) \leq -e(t)^T \left[Q - 2\tau (\|\omega(t - \tau)\|^2 + \int_{-\tau}^0 \|u(t - \tau + \sigma)\|^2 d\sigma) h_m h_m^T \right] e(t) \quad (69)$$

where $Q > 0$ satisfies $A_m^T P + P A_m = -Q$. For the non-positiveness of $\dot{V}(t)$, we need to satisfy

$$Q - 2\tau \left(\|\omega(t - \tau)\|^2 + \int_{-\tau}^0 \|u(t - \tau + \sigma)\|^2 d\sigma \right) h_m h_m^T \geq 0. \quad (70)$$

Since ω and u are dependent variables, condition (70) may not be easy to check. It is noted however that the bound on $V(t)$ is given by some bounds on ω defined at $t - \tau$ and on u defined on the whole interval $[t - 2\tau, t - \tau]$. It is shown below that this condition can be replaced by bounds on signals ω and u over the time interval $[t_0 - \tau, t_0]$ and $[t_0 - 2\tau, t_0]$, respectively.

Suppose that

$$\sup_{\xi \in [t_0 - \tau, t_0]} \|\omega(\xi)\|^2 \leq \gamma_1 \quad (71)$$

$$\sup_{\xi \in [t_0 - 2\tau, t_0]} \|u(\xi)\|^2 \leq \gamma_2 \quad (72)$$

for some $\gamma_1 > 0, \gamma_2 > 0$ and a $\tau_1 > 0$ is such that

$$2\tau_1 (\gamma_1 + \tau_1 \gamma_2) h_m h_m^T < Q. \quad (73)$$

Then, the following inequality is satisfied:

$$Q - 2\tau (\|\omega(\xi - \tau)\|^2 + \int_{-\tau}^0 \|u(\xi - \tau + \sigma)\|^2 d\sigma) h_m h_m^T > 0, \quad \forall \xi \in [t_0, t_0 + \tau), \forall \tau \in [0, \tau_1]. \quad (74)$$

It follows that $V(t)$ is non-increasing for $t \in [t_0, t_0 + \tau)$. Thus, we have

$$\|X_p(\xi)\| \leq \sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}} + \|X_m(\xi)\| \quad (75)$$

and hence,

$$\|\omega(\xi)\| \leq \sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}} + \|X_m(\xi)\|, \quad \forall \xi \in [t_0, t_0 + \tau). \quad (76)$$

The inequality in (76) is due to the fact that ω is a part of the state vector X_p . We also have the following inequalities as a result of non-increasing Lyapunov functional:

$$\|\tilde{\theta}(\xi)\|^2 \leq V(t_0), \quad (77)$$

$$\int_{-\tau}^0 \tilde{\phi}(\xi, \sigma)^2 d\sigma \leq V(t_0), \quad \forall \xi \in [t_0, t_0 + \tau]. \quad (78)$$

To simplify the notation, we define

$$I_0 \triangleq \max \left(\sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}} + \sup_{[t_0, t_0 + \tau]} \|X_m(\xi)\|, V(t_0), \sqrt{V(t_0)} \right). \quad (79)$$

An upper bound on the control signal $u(t)$ for $t \in [t_0, t_0 + \tau]$ can be derived by using Lemma 1. In particular, setting $t'_i = t_0$, $t'_j = t_0 + \tau$, $c_0^2 = V(t_0)$ and using (40), (72) and (77)–(79), we obtain that

$$|u(\xi)| \leq 2 \left(\bar{f} + \left(\int_{-\tau}^0 (u(t_0 + \sigma))^2 d\sigma \right)^{1/2} I_0 \right) e^{l_0 \tau} \quad \forall \xi \in [t_0, t_0 + \tau], \quad (80)$$

where \bar{f} depends only on I_0 . For simplicity, we will define $g(\gamma_2, I_0, \tau) \triangleq 2(\bar{f} + \gamma_2 I_0 \sqrt{\tau}) e^{l_0 \tau}$ and rewrite (80) as

$$|u(\xi)| \leq g(\gamma_2, I_0, \tau), \quad \forall \xi \in [t_0, t_0 + \tau]. \quad (81)$$

Step 2. A delay value, τ , is found that leads to a non-increasing \dot{V} over $[t_0, t_0 + 2\tau]$, which in turn shows that X_p is bounded over the same interval.

Consider a delay value of $\tau_2 > 0$ that satisfies

$$2\tau_2 (I_0^2 + (\max(\gamma_2, g(\gamma_2, I_0, \tau_2)))^2 \tau_2) h_m h_m^T < Q. \quad (82)$$

For $\bar{\tau}_2 = \min(\tau_1, \tau_2)$, (70) is satisfied in the interval $[t_0, t_0 + 2\tau]$, for all $\tau \in [0, \bar{\tau}_2]$. Therefore, we obtain that

$$\|X_p(\xi)\| \leq I_0, \quad \forall \xi \in [t_0, t_0 + 2\tau]. \quad (83)$$

Step 3. It is shown that the bound on the control signal u over the time interval $[t_0, t_0 + \tau]$ depends only on A_p, b_p, T, I_0 and τ . The proof given in this section is similar to the one given in Narendra, Annaswamy, and Singh (1985), Lemma 5.

Let $\Psi_u = \{t \mid |u(t)| = \sup_{\sigma \leq t} |u(\sigma)|\}$. Let $[t_i - \tau, t_i - \tau + T] \subset \Psi_u \subset [t_0, t_0 + \tau)$. Defining $z(t) = u(t - \tau)$, we can solve (23) as

$$x_p(t_i + T) = x_p(t_i) e^{A_p T} + \int_{t_i}^{t_i + T} e^{A_p(T+t_i-t)} b_p z(t) dt. \quad (84)$$

Positive constants c_6 and c_7 exists such that $\|e^{A_p T}\| \leq c_6$ and $\| \int_0^T e^{A_p \sigma} b_p d\sigma \| \geq c_7$ and

$$\|x_p(t_i + T)\| \geq |c_7 z(t_i)| - c_6 \|x_p(t_i)\|. \quad (85)$$

It is noted that we obtain (85) by selecting T such that the terms of the vector $e^{A_p \sigma} b_p$ do not change sign for $\sigma \in [0, T]$. In addition, it is noted that once T is selected properly, the inequality (85) is satisfied for any time interval $[t_j, t_j + T]$ as soon as $[t_j - \tau, t_j + T - \tau] \subset \Psi_u \subset [t_0, t_0 + \tau)$ since the constants c_6 and c_7 are determined only by the size of T .

From (85) we obtain that

$$\|x_p(t_i + T)\| + \|x_p(t_i)\| c_6 \geq c_7 |z(t_i)| \quad (86)$$

or

$$|u(t_i - \tau)| \leq \frac{c_6}{c_7} \|x_p(t_i)\| + \frac{1}{c_7} \|x_p(t_i + T)\|. \quad (87)$$

Since x_p is a part of the state vector X_p , $\|x_p(t)\| \leq \|X_p(t)\|$, for any t . In addition, we find using (83) that $\|x_p(t)\| \leq I_0$ in the time interval $[t_0, t_0 + 2\tau)$. Therefore, (87) can be simplified as

$$|u(t_i - \tau)| \leq \frac{(1 + c_6)}{c_7} I_0. \quad (88)$$

Using Lemma 1 by setting $t'_i = t_i - \tau$, $t'_j = t_i - \tau + T$ and $c_0^2 = V(t_0)$, using (79) and noting that $(\int_{-\tau}^0 (u(t_i - \tau + \sigma))^2 d\sigma)^{1/2} = c_1$, we obtain that

$$|u(t_i + T - \tau)| \leq 2 \left(\bar{f} + I_0 \sqrt{\int_{-\tau}^0 (u(t_i - \tau + \sigma))^2 d\sigma} \right) e^{l_0 T}. \quad (89)$$

Since $|u(t)| = \sup_{\sigma \leq t} |u(\sigma)|$ over $t \in [t_i - \tau, t_i + T - \tau]$, (89) can be simplified as

$$|u(t_i + T - \tau)| \leq 2 \left(\bar{f} + u(t_i - \tau) I_0 \sqrt{\tau} \right) e^{l_0 T}. \quad (90)$$

Using (88), (90) can be simplified as

$$|u(t_i + T - \tau)| \leq 2 \left(\bar{f} + \frac{\sqrt{\tau}(1 + c_6)}{c_7} I_0^2 \right) e^{c_0 T}. \quad (91)$$

Since $|u(t)| = \sup_{\sigma \leq t} |u(\sigma)|$ over $t \in [t_i - \tau, t_i - \tau + T]$ and t_i is any arbitrary time instant in $\Phi_u \cap [t_0, t_0 + \tau)$, it follows that $\forall t \in \Psi_u \cap [t_0, t_0 + \tau)$,

$$|u(t)| \leq 2 \left(\bar{f} + \frac{\sqrt{\tau}(1 + c_6)}{c_7} I_0^2 \right) e^{l_0 T}. \quad (92)$$

When $t \notin \Psi_u$, the inequality (92) is strengthened further. Hence, it follows that the inequality holds for the whole interval $[t_0, t_0 + \tau)$. It is noted that the right-hand side of the inequality is a function of A_p, b_p, T and I_0 . For simplicity, we are going to drop the dependence on A_p, b_p and T and represents the inequality as in the following, which simplifies the notation:

$$|u(t)| \leq U(I_0), \quad t \in [t_0, t_0 + \tau) \quad (93)$$

where U is a continuous function.

Step 4. Finally, a uniform upper bound τ^* is given for the time-delay τ such that I is satisfied.

Let the delay value τ_3 satisfy the following inequality:

$$2\tau_3 [(\max(U(I_0), g(U(I_0), I_0, \tau_3)))^2 \tau_3 + I_0^2] h_m h_m^T < Q. \quad (94)$$

For $\tau^* = \min(\bar{\tau}_2, \tau_3)$, the following inequalities hold:

$$\|X_p(\xi)\| \leq I_0, \quad |u(\xi)| \leq U(I_0), \quad \forall \xi \in [t_0, t_0 + \tau), \quad \forall \tau \in [0, \tau^*]. \quad (95)$$

This completes the proof of I.

It is noted that, in Steps 1–3, I is proved for a delay value $\tau \in [0, \bar{\tau}_2]$. Finally in Step 4, a new upper bound, τ^* , is given for the allowable time-delay τ . The need for the introduction of this new upper bound will be clear in the next sub-section, where we prove II.

3.3.2. II. If S is true for k , then it is true for $k + 1$

Assume that

$$\|X_p(\xi)\| \leq I_0, \quad |u(\xi)| \leq U(I_0) \quad \xi \in [t_0, t_0 + k\tau]. \quad (96)$$

Then, using (94) we conclude that the Lyapunov function is non-increasing in the time interval $[t_0+k\tau, t_0+(k+1)\tau]$ since $\tau \leq \tau^* \leq \tau_3$. This means that $\|X_p(t)\| \leq I_0$, and, using Lemma 1, $|u(t)| \leq g(U(I_0), I_0, \tau)$ for $t \in [t_0+k\tau, t_0+(k+1)\tau]$. Again, using (94) and the fact that $\tau \leq \tau^* \leq \tau_3$, we conclude that the Lyapunov function is non-increasing in the time interval $[t_0+(k+1)\tau, t_0+(k+2)\tau]$. Hence, $\|X_p(t)\| \leq I_0$ for $t \in [t_0+(k+1)\tau, t_0+(k+2)\tau]$. But this means that $|u(t)| \leq U(I_0)$ for $t \in [t_0+k\tau, t_0+(k+1)\tau]$ using the same procedure as in Step 3. Therefore, we have that

$$\|X_p(\xi)\| \leq I_0, \quad |u(\xi)| \leq U(I_0) \quad \xi \in [t_0, t_0+(k+1)\tau]. \quad (97)$$

This completes the proof of II.

Therefore, we proved the boundedness of the signals in the system, using induction. The second derivative of the Lyapunov functional includes these signals as well as their derivatives which can be shown to be bounded using the adaptation laws (47). Hence, the requirements of Barbalat's lemma are satisfied, which completes the stability proof. It is noted that the region of stability depends on the time-delay and therefore the stability is semi-global. \square

Remark 1. We make a few comments about the development of the overall control architecture. The first step in this architecture is the design of a fixed, non-adaptive controller. This was carried out by beginning with a control design (Fig. 1) quite similar to an SP-controller. The resulting controller is rational but non-causal, and therefore is transformed into a causal, infinite-dimensional controller (36). The second step of this architecture is the development of adaptation laws for time-varying parameters which estimate the unknown control parameters for the case when the plant has unknown parameters. This is carried out in a similar manner to that proposed in Niculescu and Annaswamy (2003).

Remark 2. A Lyapunov–Krasovskii functional (68) is used to begin the stability proof, which is similar to the procedure in Niculescu and Annaswamy (2003). Where this paper differs from Niculescu and Annaswamy (2003) is that this functional, in our case, does not depend on the infinite-dimensional state introduced by the control input history of one delay interval. The structure of the control input is then exploited to show the boundedness of all state variables. Growth rate conditions, similar to those used in Narendra and Annaswamy (2005), are used in this regard. This procedure helps us remove the assumption of plant poles having multiplicity one.

A few remarks about the nature of the proof are in order.

Remark 3. Induction is the tool that is used for the stability proof. In the first part, we showed that all system states are bounded in the first delay interval given that the delay value is bounded by τ^* . This is achieved by using the Lyapunov–Krasovskii functional (68). It is noted that in general, Lyapunov–Krasovskii functionals include the infinite-dimensional state introduced by one delay interval time history of the control signal to satisfy the positivity requirement. However, (68) does not include this state and therefore additional tools were needed to be used to show the boundedness of the control signal. In particular, we used signal growth rate conditions (Narendra & Annaswamy, 2005), which simply gives rules that define the relationship between the states of an LTI system and its outputs. Using this relationship, we showed in Lemma 1 that we can put an upper bound on the control signal at time t in terms of system state values during the time interval $[t-T, t)$, for any T . This lemma is used in Part I as well as in Part II of the stability proof, where we showed that given all the states are bounded for the k th interval, the states for the next, $(k+1)$ st, interval are also bounded.

Remark 4. Consider a real SISO plant which is modeled with transfer function $P(s)e^{-\tau s}$ and with an input–output pair u, y .

The classical Smith Predictor block transfer function for this plant is $\{P(s) - P(s)e^{-\tau s}\}$. This block “predicts” the present and τ s in to the future output of the real plant and simply takes the difference between these two predictions. In the implementation, the real plant output, $y(t)$, is also measured and fed back. In the end, the total feedback consists of mainly two parts: $P(s)u(t)$, the future prediction and $\{P(s)e^{-\tau s}u(t) - y(t)\}$, the error between the prediction and the real signal. As we mentioned in the introduction, this structure needs to be modified for unstable plants (for example using finite spectrum assignment). Since the adaptive controller we propose uses the finite spectrum assignment controller as the model matching controller, it also contains the prediction and error correction terms although not explicitly as in the Smith Predictor.

4. Experimental results

Here we briefly present experimental results that are obtained from the application of the APC to fuel-to-air ratio (FAR) control problem in gasoline engines. The details can be found in Yildiz et al. (2008b).

In FAR control, the objective is to maintain the in-cylinder fuel-to-air ratio at a prescribed set point, determined primarily by the state of the Three-Way Catalyst (TWC), so that the pollutants in the exhaust are removed with the highest efficiency. The main components of the FAR dynamics are the wall wetting dynamics, gas mixing dynamics, sensor dynamics and the time-delay. The time-delay in the system comprises two basic components: the time it takes from the fuel injection calculation to exhaust gas exiting the cylinders and the time it takes for the exhaust gases to reach the UEGO sensor location. For the purpose of fuel-to-air ratio feedback controller design, a standard approach (Guzzella & Onder, 2004; Zhang, Grigoriadis, Franchek, & Makki, 2007) of assuming a plant model consisting of a first order transfer function plus time-delay in series is followed:

$$G(s) = \frac{1}{\tau_m s + 1} e^{-\tau s} \quad (98)$$

where, τ_m is the system time constant, which may be uncertain due to changing operating conditions, and τ is the total time-delay. The use of simple models, such as (98), can be also advantageous from the standpoint of simplifying software implementation and calibration effort. Time-delay is experimentally identified from fuel step responses. An estimate of the delay as a function of engine speed and load (airflow) is incorporated within the engine control strategy. Typical values for τ and τ_m are 0.4 and 0.45, respectively, during warm, idle operation. These values change as the engine speed and load varies. The controller must handle the time-delay and the uncertainty in the time constant. In addition, the disturbances due to canister vapor purging and due to inaccuracies in air charge estimation and wall wetting compensation, must be rejected. Moreover, since (98) is a simplified model of more complex dynamics, the success of our adaptive controller in the experiments is an indirect corroboration of the robustness of the controller to the unmodeled dynamics. In addition, given that the controller is designed for a constant known delay while the actual time-delay in the experiments varies with the engine speed and load, the experimental results are a corroboration of the robustness of the controller to inaccurate delay measurements.

The purpose of our FAR control experiments was to compare the performances of the APC and the baseline controller, while emulating canister vapor purge disturbance rejection tests. The baseline controller is the existing controller in the development vehicle, which is essentially a gain scheduled PID controller, without any explicit delay compensation. These experiments were conducted with the test vehicle idling at different speeds.

The important success measure (SM) for the FAR control loop is the error integral, formulated as

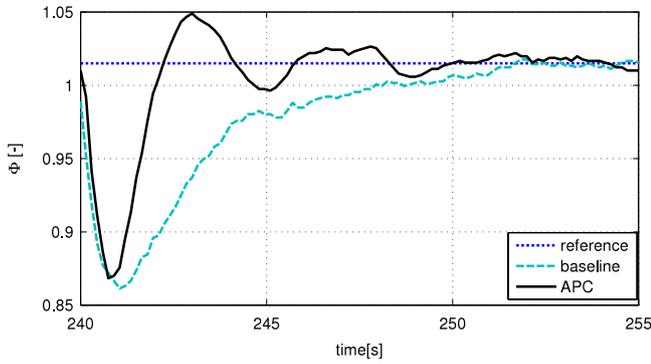


Fig. 3. Comparison of baseline controller with APC for purge disturbance rejection at 700 rpm.

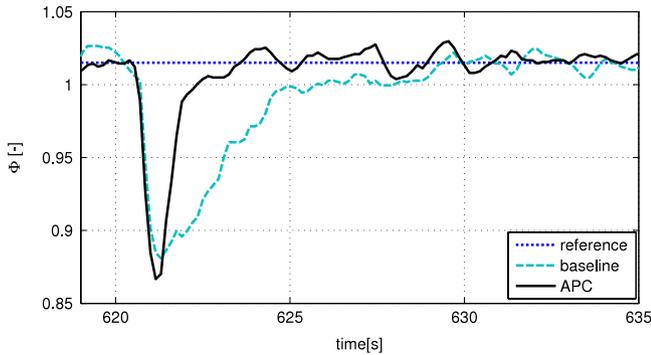


Fig. 4. Comparison of baseline controller with APC for purge disturbance rejection at 1000 rpm, with $c = 1$.

$$SM = \frac{1}{k} \sum_{i=1}^k \left| \int_{t_i}^{t_i+\Delta_i} e_1(\eta) d\eta \right| \quad (99)$$

where t_i is the time instant of the i -th disturbance hit and Δ_i is the duration/settling time of the transient caused by the disturbance hit. This metric reflects how much of the TWC oxygen storage capacity is used to compensate for the deviations in the fuel-to-air ratio, and is used as the SM for evaluating the performances of the baseline and the proposed adaptive controller.

The test started with the engine speed at 700 rpm. At 300 s, the engine speed increased to 1000 rpm and at 600 s it decreased back to 700 rpm. Every 20 s the fuel injector gains were changed to emulate the purge disturbance. Overall, the performance of the APC was 70% better, in terms of SM, than the baseline controller during the test which lasted 15 min. Fig. 3 shows a time window from the test where the engine speed was 700 rpm. The APC performs considerably better than the baseline controller as its features enable it to better account for the delay and achieve faster response. It is noted that the response of the APC is faster compared to the baseline controller and small amplitude oscillations are not detrimental as they are handled by the TWC. The error integral is an indicator of the used oxygen storage which essentially integrates out and cancels these deviations.

Fig. 4 shows how the equivalence ratio changes during the same test but now the engine speed is 1000 rpm. Again, the performance of the APC is better than that of the baseline controller.

Remark 5. We have also applied our control design approach to the idle speed control regulation in gasoline engines where the intake-to-torque production time-delay limits the achievable closed-loop performance. See Yildiz et al. (2007) for the results and favorable comparison with the existing controller in the development vehicle.

5. Summary

In this paper, we proposed an Adaptive Posicast Controller (APC) for time-delay systems. The controller is based on the Smith Predictor and finite spectrum assignment controller and is modified so as to accommodate parametric uncertainties in the plant dynamics. We first proposed a model matching controller for time-delay systems with known time-delay, where, a non-causal controller is designed first, for the rational part of the plant dynamics. Then, by using the known delay value, the future prediction of the system states are calculated and used to show that the non-causal controller is equivalent to a causal controller. Using this equivalence, we derived the closed-loop error dynamics for the APC and showed the stability of the overall system by defining a suitable Lyapunov–Krasovskii functional.

The simplicity of the APC makes it a good candidate for industrial applications where adaptation is needed. To verify this, experimental results that show the success of the APC in an important automotive control problem were presented. These implementation results show that the APC can considerably outperform the existing controller in the test vehicles.

The Adaptive Posicast Controller presented here is predicated on several assumptions, some of whom can be relaxed in a straight forward manner. The first is that the plant relative degree needs to be one or two. This assumption can be relaxed by using high order tuners for parameter adaptation, very similar to the approach used in Evesque, Annaswamy, Niculescu, and Dowling (2003). The second is the effect of disturbances and unmodeled dynamics, both of which are assumed to be absent in this paper. Similar to Niculescu and Annaswamy (2003), these effects can be accommodated by suitably modifying the adaptive law. The third assumption that zeros need to be stable is similar to what is observed in most of the adaptive control literature. The fourth is the assumption that the delay needs to be known. In a manner similar to Niculescu and Annaswamy (2003) (and the references therein), robustness to perturbations in the delay may be shown for the model matching part of the adaptive controller in this paper as well. It should be noted however, that the delay is fixed and needs to be chosen as its nominal value. A possible alternative is the result reported by Krstic and coworkers (Bresch-Pietri & Krstic, 2009) where adaptation to the delay is proposed using infinite-dimensional tools. This seems to be a highly promising direction of future research.

Acknowledgements

This work was supported through the Ford-MIT Alliance Initiative. The authors would like to acknowledge Dr. Davor Hrovat of Ford Motor Company for his support and encouragement during this project.

Appendix. The bound on the series S

Defining $a \triangleq c_0^2(t - t'_i)$ and $S' \triangleq \sum_{n=1}^{\infty} (\frac{a^n}{n!})^{1/2}$, we obtain that

$$S = (\bar{f} + c_0 c_1) S'. \quad (A.1)$$

Summing up the odd terms of S' we have that

$$\begin{aligned} S'_o &= \sqrt{a} + \sqrt{\frac{a^3}{3!}} + \sqrt{\frac{a^5}{5!}} + \dots \\ &= \sqrt{a} + a\sqrt{\frac{a}{3!}} + \frac{a^2}{2!}\sqrt{\frac{a}{30}} + \dots \end{aligned} \quad (A.2)$$

If $a < 1$, then we obtain that

$$S'_o \leq 1 + a + \frac{a^2}{2!} + \dots = e^a. \quad (A.3)$$

If $a > 1$, then we obtain that

$$\begin{aligned} S'_o &\leq a + \frac{a^2}{2!} \frac{1}{\sqrt{1.5}} + \frac{a^3}{3!} \frac{1}{\sqrt{10/3}} + \dots \\ &\leq 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots \\ &= e^a. \end{aligned} \quad (\text{A.4})$$

Summing up the even terms, we have

$$\begin{aligned} S'_e &= \frac{a}{\sqrt{2!}} + \frac{a^2}{\sqrt{4!}} + \frac{a^3}{\sqrt{6!}} + \dots \\ &\leq 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots \\ &= e^a. \end{aligned} \quad (\text{A.5})$$

Using (A.1) and (A.3)–(A.5) we obtain that

$$\begin{aligned} S &= (\bar{f} + c_0 c_1) (S'_o + S'_e) \\ &\leq 2(\bar{f} + c_0 c_1) e^a. \end{aligned} \quad (\text{A.6})$$

Finally, substituting the definition of a in (A.6) we have

$$S \leq 2(\bar{f} + c_0 c_1) e^{c_0^2(t-t'_i)}. \quad (\text{A.7})$$

This completes the proof.

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Yildiray Yildiz is a scientist at UCSC, NASA Ames Research Center, Moffett Field, California. He received his Ph.D. degree in mechanical engineering from Massachusetts Institute of Technology, Boston, in 2009, M.S. degree in mechatronics from Sabanci University, Istanbul, Turkey, in 2004, and B.S. degree in mechanical engineering from Middle East Technical University, Ankara, Turkey, in 2002. His research interests lie in the intersection of control theory and its applications to automotive and aerospace systems. Specifically, he is interested in adaptive control, time-delay systems, automotive powertrain control, aircraft control and handling qualities and pilot induced oscillations. His research on adaptive control of time-delay systems and applications to automotive control was recognized by ASME Dynamic Systems & Control Division with a student best paper award in 2008. Dr. Yildiz is a member of IEEE, ASME and AIAA.

Anuradha Annaswamy received the Ph.D. degree in electrical engineering from Yale University in 1985. She has been a member of the faculty at Yale, Boston University, and MIT, where currently she is the director of the Active Adaptive Control Laboratory and a Senior Research Scientist in the Department of Mechanical Engineering. Her research interests pertain to adaptive control theory and applications to aerospace and automotive control, neural networks, active control of noise in thermo-fluid systems, active emission control, and active adaptive control of autonomous vehicles in air and undersea. She has authored

numerous journal and conference papers and a graduate textbook on adaptive systems.

Dr. Annaswamy has received several awards including the Alfred Hay Medal from the Indian Institute of Science in 1977, the Stennard Fellowship from Yale University in 1980, the IBM postdoctoral fellowship in 1985, the George Axelby Outstanding Paper award from IEEE Control Systems Society in 1988, the Presidential Young Investigator award from the National Science Foundation in 1991, the Hans Fisher Senior Fellowship from the Institute for Advanced Study at Technical University of Munich in 2008, and the Donald Groen Julius Prize for 2008 from the Institute of Mechanical Engineers. Dr. Annaswamy is a Fellow of the IEEE and a member of AIAA.

Dr. Annaswamy has served as an Associate Editor of several journals including the IEEE Transactions on Automatic Control and Automatica. She has also served as the General Chair of the 2008 ACC, as the Program Chair of the 2003 ACC, and on the program committee of several international conferences. She was selected as a Distinguished Lecturer and elected as a member of the Board of Governors of the IEEE Control Systems Society in 2002 and 2009, respectively.

